International Journal of Theoretical Physics, Vol. 15, No. 11 (1976), pp. 841–845

A Strong Interaction Theory with Internal Coordinates

F. C. HOH

4618 Spruce Street, Philadelphia, Pennsylvania 19139

Received: 16 October 1975

Abstract

Quantum theory and SU_3 classification of hadrons are partially unified and are extended to produce a single formalism. The theory accounts for the possibility that a number of different hadrons can be exchanged between two quarks without explicitly assuming their masses and quantum numbers. Quantization of the theory leads to the conclusion that a quark has spin $\frac{1}{2}$ but obeys Bose statistics and naturally accounts for the relation between spin and statistics for the baryon decuplet.

Quantum theory, essentially formulated about half a century ago, has proved to be inadequate in accounting for hadron behavior. The later approach, the SU_3 scheme (Gell-Mann, 1962; Ne'eman, 1961) and quark hypothesis (Gell-Mann, 1964; Zweig, 1964), has, however, been highly successful in classifying hadrons. These two approaches have recently been partially united and extended into one formalism (Hol, 1975a). When applied to pseudoscalar mesons, the Gell-Mann-Okubo formula for these mesons was derived with the coefficients determined by given relations (Hoh, 1975b). The purpose of this note is to give a brief outline of the theory, to point out that the present theory does not in principle need as input parameters the masses and quantum numbers of the particles involved as does quantum theory, and to account for the relation between spin and statistics of hadrons, e.g., members of the baryon decuplet, in a natural way.

The present starting point is the Dirac equation for a free particle in spinor form (van der Waerden, 1929; Laporte and Uhlenbeck, 1931):

$$\begin{aligned} i\partial_\nu{}^{\dot\sigma}\psi_{\dot\sigma}(x) &= -m\chi_\nu(x) \\ i\partial_{\dot\tau}{}^{\nu}\chi_\nu(x) &= m\psi_{\dot\tau}(x) \end{aligned} \tag{1}$$

For a free quark, this equation is generalized to

$$\begin{aligned} i\partial_\nu{}^{\dot\sigma}\psi_{\dot\sigma}(x)\xi^a(z) &= \partial_b{}^a\chi_\nu(x)\xi^b(z) \\ i\partial_{\dot\tau}{}^{\nu}\chi_\nu(x)\xi^b(z) &= -\partial_c{}^b\psi_{\dot\tau}(x)\xi^c(z) \end{aligned} \tag{2}$$

which, in bispinor form, has the form

$$i\gamma^\mu \partial_\mu \psi(x)\xi^a(z) + \partial_b{}^a \psi(x)\xi^b(z) = 0 \tag{3}$$

Here, a, b, and c each runs from 1 to 3 and $z = (z^a, z_a) = (z', z^2, z^3, z_1, z_2, z_3)$. z^1, z^2, and z^3 are three complex coordinates in a complex three-dimensional space M_3 and are called internal coordinates. Further, $z^{a*} = z_a$, $\partial^a = \partial/\partial z_a$, and $\partial_b{}^a = \partial^2/\partial z_a \partial z^b$. z^a can be transformed to spherical coordinates (Bég and Ruegg, 1965):

$$\begin{aligned} z^1 &= r \sin \vartheta \cos \xi e^{i\varphi_1} \\ z^2 &= r \sin \vartheta \sin \xi e^{i\varphi_2} \\ z^3 &= r \cos \vartheta e^{i\varphi_3} \end{aligned} \tag{4}$$

A volume element in M_3 can be written as

$$dz_1 dz^1 dz_2 dz^2 dz_3 dz^3 = 8r^5 \cos\vartheta \sin^3\vartheta \cos\xi \sin\xi \, dr\, d\vartheta\, d\xi\, d\varphi_1 d\varphi_2 d\varphi_3 \tag{5}$$

Bég and Ruegg (1965) introduced a set of orthogonal functions in M_3. When normalized, it reads

$$Y^{pq}_{YII_3}(\vartheta, \xi, \varphi_1, \varphi_2, \varphi_3) = e^{i\delta} \left[\frac{(2I+1)(p+q+2)}{2\pi^3}\right]^{1/2} \frac{1}{\sin\vartheta}$$

$$\times\, d^{(p+q+1)/2}_{(p-q-3Y+6I+3)/6,\,(p-q-3Y-6I-3)/6}(2\vartheta)\, d^I_{(p-q)/3+Y/2,\, I_3}(2\xi)$$

$$\times \exp\{i[\tfrac{1}{3}(p-q)(\varphi_1+\varphi_2+\varphi_3) + \tfrac{1}{2}Y(\varphi_1+\varphi_2-2\varphi_3) + I_3(\varphi_1-\varphi_2)]\} \tag{6}$$

Where δ is a phase factor which may depend upon p, q, Y, I, and I_3 which in turn are discrete constants with the usual quark theory interpretation.

$\xi^a(z)$ has been expanded in a series of the Y's in (6) for the case of a one-quark system. Keeping the lowest order in the expansion only, one has

$$\xi^1 = q_1(r) Y^{10}_{\frac{1}{3}\frac{1}{2}\frac{1}{2}}, \qquad \xi^2 = q_2(r) Y^{10}_{\frac{1}{3}\frac{1}{2}-\frac{1}{2}}, \qquad \xi^3 = q_3(r) Y^{10}_{-\frac{2}{3}00} \tag{7}$$

Separating (3) into space-time and internal parts and making use of (7), one obtains

$$q_1 = q_2 = q_3 = q(r) = \text{const}\, \frac{1}{r^2} J_3(\sqrt{2K_s} r) \tag{8}$$

where K_s can be interpreted as the indeterminate mass of a free quark.

It was proposed (Hoh, 1975a) that a quark interacts with another quark in space-time as well as in M_3 and that, when interactions are included, (3) was generalized to

$$\begin{aligned} i\gamma^\mu \partial_\mu \psi(x)\xi^a(z) + \partial_b{}^a \psi(x)\xi^b(z) &= [\gamma_5 U_p(x) + \gamma^\mu U_\mu(x) + \gamma^\mu A_\mu(x)]\, \psi(x)\xi^a(z) \\ &+ \{[\tau(z) + G_{mp}(z)(\lambda_8)_b{}^a] + [\omega_b{}^a(z)(\lambda_8)_b{}^a] + \\ &+ G_{em}(z) Q_b{}^a\}\psi(x)\xi^b(z) \end{aligned} \tag{9}$$

Where $2Q = \lambda_3 + \lambda_8/\sqrt{3}$ and the λ's are the Gell-Mann matrices. Further, U_p, U_μ, and A_μ are pseudoscalar, vector, and electromagnetic interaction functions, respectively, and τ, ω, $G_{mp} + G_{mv}$, and G_{em} are singlet, nonet, hypercharge, and electromagnetic interactions, respectively, in the internal space M_3. The internal and space-time interaction functions were connected via similarly generalized Klein-Gordon equations:

$$(\Box - \partial_b{}^a\partial_a{}^b)U_\mu(x)(\omega_b{}^a(z) + G_{mv}(z)(\lambda_8)_b{}^a) = \mu_v\bar{\chi}(x)\gamma_\mu\chi(x)[\omega_b{}^a(z) + G_{mv}(z)(\lambda_8)_b{}^a] - [\mu_0\zeta^a(z)\zeta_b(z) + \mu_8\zeta^c(z)\zeta_c(z)(\lambda_8)_b{}^a]\,U_\mu(x) \qquad (10)$$

$$\Box A_\mu(x) = \mu_\alpha\bar{\chi}(x)\gamma_\mu\chi(x) \qquad (11)$$

$$\partial_b{}^a\partial_a{}^b G_{em}(z) = \mu_Q\zeta^c(z)\zeta_c(z) \qquad (12)$$

and an equation for $U_p(\tau + G_{mp}\lambda_8)$ similar to (10). The m^2 term in the Klein-Gordon equation was replaced by $\partial_b{}^a\partial_a{}^b$ like m was replaced by $-\partial_b{}^a$ in (2). $\chi(x)\zeta^a(z)$ is the wave function of the other interacting quark, corresponding to $\psi(x)\xi^a(z)$, and the μ's are interaction parameters supposedly known.

If quantum theory is applied to such a case, the quark mass as well as the masses of the exchanged particles between the two interacting quarks must be introduced as fixed parameters. The present theory has the advantage over quantum theory in that the masses have been replaced by operators and need in principle not be assumed. The masses may be associated with eigenvalue of the operators. (10) can be separated into space-time and internal parts:

$$(\Box - m_v{}^2)U_\mu(x) = \mu_v\bar{\chi}(x)\gamma_\mu\chi(x) \qquad (13)$$

$$(\partial_b{}^a\partial_a{}^b - m_v{}^2)[\omega_b{}^a(z) + G_{mv}(z)(\lambda_8)_b{}^a] = \mu_0\zeta^a(z)\zeta_b(z) + \mu_8\zeta^c(z)\zeta_c(z)(\lambda_8)_b{}^a \qquad (14)$$

Here, the separation constant $m_v{}^2$ is not fixed but can vary or possibly be quantized like the angular momentum separation constant l in quantum theory. m_v can be interpreted as the mass of the exchanged particle or the masses of such particles if m_v can assume different values in an interaction. Equation (9) can be similarly separated to give

$$(i\gamma^\mu\partial_\mu - m_q)\psi(x) = [\gamma_5 U_p(x) + \gamma^\mu U_\mu(x) + \gamma^\mu A_\mu(x)]\,\psi(x) \qquad (15)$$

$$\partial_b{}^a\xi^b(z) + m_q\xi^a(z) = \{\tau(z) + \omega_b{}^a(z) + [G_{mp}(z) + G_{mv}(z)]\,(\lambda_8)_b{}^a + G_{em}(z)Q_b{}^a\}\xi^b(z) \qquad (16)$$

It was assumed (Hoh, 1975b) that an internal meson wave function $\xi_b{}^a(z)$ consists of a zero-order SU_3 symmetry-preserving part $(\xi_0)_b{}^a(z)$ and a first-order part $(\xi_1)_b{}^a(z)$ dependent upon a hypercharge interaction term $\propto \lambda_8$. Then, the zero order part was expanded:

$$(\xi_0)_b{}^a(z) = \sum_{p,\,q,\,Y,I,\,I_3} [(\xi_0)_b{}^a(z)]^{pq}_{YII_3} \qquad (17)$$

$$[(\xi_0)_b{}^a(z)]^{pq}_{YII_3} = g(p, q, Y, I, I_3, r)Y^{pq}_{YII_3} + \sum_{p'q'} f_{p'+q'}(p'-q', Y, I, I_3, r)\mathbf{Y}^{p'q'}_{YII_3}(p, q, \vartheta, \xi, \varphi_1, \varphi_2, \varphi_3)\boldsymbol{\lambda}_b{}^a \tag{18}$$

Where $p + q - 2 \leqslant p' + q' \leqslant p + q + 2$ and the **Y**'s are 8-vector spherical harmonics defined in the internal space and correspond to the usual spherical harmonics in space. The term with $p = q = 0$ in (17) was associated with an SU_3 singlet meson and the term with $p = q = 1$ with eight mesons belonging to an SU_3 octet.

If $\omega_b{}^a + G_{mv}(\lambda_8)_b{}^a$ in (10) is similarly treated like $\xi_b{}^a$ above, then one sees that the exchanged particle can in principle be a combination of nine mesons when $p = q \leqslant 1$. For $p = q \leqslant 2$, the combination is enlarged by a 27plet. For $p = q \leqslant 3$, it is further augmented by a set of bound baryon–antibaryon pairs. Such exchanged particles, like the mesons, can possibly be emitted. The bound baryon–antibaryon pairs may perhaps be dissociated to produce free baryons and associated antibaryons. When applied to such cases, quantum theory may assume a rather bulky form involving a large number of mass and interaction parameters. The present formalism has the formal advantage over quantum theory in that the large number of different kinds of exchanged particles is formally represented in a simple and more natural way.

A more appropriate description of quark–quark interaction than that given above according to the present view is a Bethe–Salpeter equation generalized to properly include internal coordinates. Such a generalized Bethe–Salpeter equation in the ladder approximation for a quark–antiquark pair has been presented (Hoh, 1975a) and treated (Hoh, 1975b). Under a consistent set of approximations, the Gell-Mann–Okubo formula for pseudoscalar mesons was reproduced with the coefficients given explicitly in terms of eigenvalues and eigenfunctions obeying given zero-order equations. Mixing between meson nonet states was shown to be associated with removal of possible degeneracy among zero-order eigenfunctions.

Quantization of the present theory of strong interactions is assumed to take place in both space-time and the internal space. For a quark with a wave function of $\psi(x)\xi^a(z)$, as in (9), the following anticommutation relations for $\psi(x)$ and $\xi^a(z)$ are assumed:

$$\{\psi_\nu(\mathbf{x}, t), \psi_\lambda(\mathbf{x}', t)\} = \{\psi_\nu^*(\mathbf{x}, t), \psi_\lambda^*(\mathbf{x}', t)\} = 0 \tag{19}$$

$$\{\psi_\nu(\mathbf{x}, t), \psi_\lambda^*(\mathbf{x}', t)\} = \delta_{\nu\lambda}\delta(\mathbf{x} - \mathbf{x}') \tag{20}$$

$$\{\xi^a(z), \xi^b(z')\} = \{\xi_a(z), \xi_b(z')\} = 0 \tag{21}$$

$$\{\xi^a(z), \xi_b(z')\} = \delta_b{}^a\delta(z - z') \tag{22}$$

where ν and λ each runs from 1 to 4 and denotes the four $\psi(x)$ components. Further, any component of $\xi^a(z)$ is assumed to commute with any component of $\psi(x)$. The quark wave function $\psi(x)\xi^a(z)$ therefore satisfies the following commutation relations:

$$[\psi_\nu(\mathbf{x}, t)\xi^a(z), \psi_\lambda(\mathbf{x}', t)\xi^b(z')] = [\psi_\nu^*(\mathbf{x}, t)\xi_a(z), \psi_\lambda^*(\mathbf{x}', t)\xi_b(z')] = 0 \quad (23)$$

$$[\psi_\nu(\mathbf{x}, t)\xi^a(z), \psi_\lambda^*(\mathbf{x}', t)\xi_b(z')] = \delta_{\nu\lambda}\delta(\mathbf{x} - \mathbf{x}')\delta_b{}^a\delta(z - z') + \psi_\lambda^*(\mathbf{x}', t)\psi_\nu(\mathbf{x}, t)\delta_b{}^a\delta(z - z') + \delta_{\nu\lambda}\delta(\mathbf{x} - \mathbf{x}')\xi_b(z')\xi^a(z) \quad (24)$$

Thus, a quark is described by a spin-$\frac{1}{2}$ wave function in space time and is therefore a fermion in this sense. The quark wave function, including both space-time and internal parts, satisfies the commutation relations (23) and (24) and therefore obeys in this sense Bose statistics. A many-quark wave function is thus always symmetric under permutations of different quarks. The totally symmetric baryon decuplet has spin $\frac{3}{2}$ and obeys Fermi statistics, since, as physical particles, the internal parts of its wave functions have been integrated out. The present theory naturally accounts for the relation between spin and statistics for the baryon decuplet and makes the earlier suggestions that there are three so-called colored triplets of quarks with integral or with fractional charges unnecessary.

The internal wave function of a single quark is given by ξ^a (7). In this case, one can show that ξ^a becomes identical with z^a in (4) if $q_1 = q_2 = q_3 = r\pi^{3/2}/\sqrt{6}$. The internal wave function of a single quark, ξ^a in (7), is therefore an odd function under reflection in M_3 or under the transformation $z^a \to - z^a$. This agrees with the earlier assumption that $\xi^a(z)$ obey anticommutation relations.

References

Beg, M. A. B., and Ruegg, H. (1965). *Journal of Mathematical Physics*, **6**, 677.
Gell-Mann, M. (1962). *Physical Review*, **125**, 1067.
Gell-Mann, M. (1964). *Physics Letters*, **8**, 214.
Hoh, F. C. (1975a). *International Journal of Theoretical Physics*, **16**, 847.
Hoh, F. C. (1975b). Meson theory with internal coordinates (submitted for publication).
Laporte, O., and Uhlenbeck, G. E. (1931). *Physical Review*, **37**, 1380.
Ne'eman, Y. (1961). *Nuclear Physics*, **26**, 222.
van der Waerden, B. (1929). *Göttinger Nachrichten*, 100.
Zweig, G. (1964). CERN Reports Th. 401 and Th. 412.